The analytic solution of interfacial concentration with observed rejection ratio during dead-end membrane filtration

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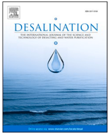

The analytic solution of interfacial concentration with observed rejection ratio during dead-end membrane filtration

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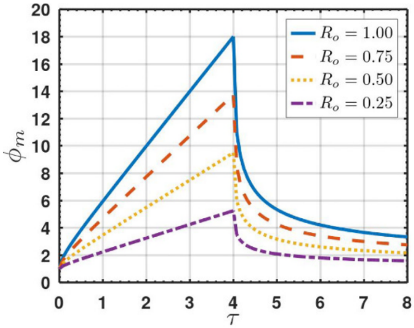
Interfacial Concentrations and Intrinsic vs. Observed Rejections



$$\phi_m(\tau \leq \tau_s) = 1 + 2R_o\left[\tau + \left(\tau + \tfrac{1}{2}\right)\mathrm{erf}\left(\sqrt{\tau}\right) + \sqrt{\frac{\tau}{\pi}}e^{-\tau}\right]$$

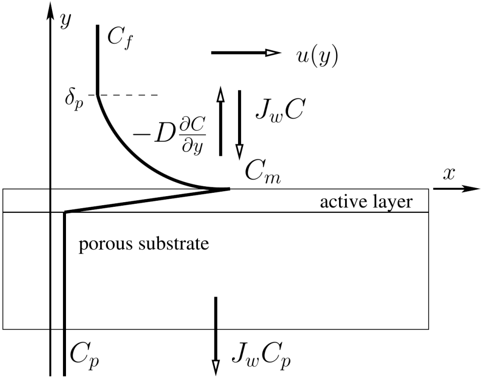
$$\phi_m(\tau_s < \tau) = 1 + \left[\phi_m(\tau_s) - 1\right]e^{\beta^2(\tau-\tau_s)}\,\mathrm{erfc}\left(\beta\sqrt{\tau - \tau_s}\right)$$

Interfacial concentration profiles for constant-flux dead-end filtration, obtained analytically, with various observed rejection ratios

$$R_i \simeq \sqrt{R_o}$$

- The intrinsic rejection ratio can be estimated as a square root of the observed rejection.

Governing Equation and Conditions



The governing equation of dead-end filtration may be written as

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial y}\left(D_0 \frac{\partial C}{\partial y} + J_w C\right) \qquad (1)$$

with initial and boundary conditions of

$$C\,(y, t=0) = C_f \qquad (2)$$

$$C\,(y \to \infty, t) = C_f \qquad (3)$$

$$D_0 \frac{\partial C_m}{\partial y} + J_w C_m(t) = J_w C_p \qquad (4)$$

where $C_m = C\,(t, y=0)$.

Non-dimensionalization

The dimensionless governing equation is

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2} + \mathrm{Pe}\, \lambda \frac{\partial \phi}{\partial \eta} \quad (5)$$

using

$$C = C_f \phi \quad (6)$$
$$y = L\eta \quad (7)$$
$$t = T\tau \quad (8)$$

and λ is a flow parameter, such as

- regular dead-end flow ($\lambda = 1$)
- pressure release ($\lambda = 0$)
- backwashing flow ($\lambda = -1$)

And, the Peclet number is defined and set as, for convenience,

$$\mathrm{Pe} = \frac{J_w L}{D_0} = 2 \quad (9)$$

Then, the parameters are estimated

$$L = D_0/2J_{w0} = 0.37\,\mathrm{mm} \quad (10)$$
$$T = 4D_0/J_{w0}^2 = 20.0\,\mathrm{min} \quad (11)$$

using

$$D = O\left(10^{-9}\right) \mathrm{m}^2/\mathrm{s} \quad (12)$$
$$J_w = 10.0 \text{ LMH} \quad (13)$$

Dimensionless Formalism

We rewrite the governing equation in a dimensionless form using η and τ, such as

$$\frac{\partial\phi}{\partial\tau}=\frac{\partial^2\phi}{\partial\eta^2}+2\lambda\frac{\partial\phi}{\partial\eta} \tag{14}$$

with the initial condition

$$\phi\left(\tau=0,\eta\right) = 1 \tag{15}$$

and boundary conditions of

$$\phi\left(\eta\to\infty,\tau\right) = 1 \tag{16}$$

$$\frac{\partial\phi_m}{\partial\eta}+2\lambda\left(\phi_m-\phi_p\right) = 0 \tag{17}$$

where

$$\phi_p=\frac{C_p}{C_f}=1-R_o \tag{18}$$

is the dimensionless permeate concentration in terms of R_o, and

$$\phi_m\left(\tau\right) = \frac{C_m\left(t\right)}{C_f} \tag{19}$$

$$\frac{\partial\phi_m}{\partial\eta} = \left[\frac{\partial\phi}{\partial\eta}\right]_{\eta=0} \tag{20}$$

are the dimensionless interfacial concentration and its derivative, repsectively.

Solution Procedure and Analytic solution, Almost!

The Laplace transform for $\phi(\tau,\eta)$ is

$$\mathcal{L}\left[\phi(\tau,\eta)\right] = \Phi(p,\eta) \quad (21)$$

$$= \int_0^\infty e^{-p\tau}\phi(\tau,\eta)\,\mathrm{d}\tau \quad (22)$$

After many hours, we obtain the particular Φ satisfying the three conditions:

$$\Phi(p,\eta) = \frac{1}{p} + 2R_o \cdot \frac{e^{-\left(1+\sqrt{1+p}\right)\eta}}{p\left(\sqrt{1+p}-1\right)} \quad (23)$$

Ready for an inverse Laplace transform!

This is it for the full concentration profile.

$$\phi(\tau,\eta) = 1 + 2R_o\mathcal{L}^{-1}\left[\frac{e^{-\left(1+\sqrt{1+p}\right)\eta}}{p\left(\sqrt{1+p}-1\right)}\right] \quad (24)$$

But, too difficult. Let's find an easy shortcut, i.e., the interfacial concentation at $\eta = 0$.

$$\phi_m(\tau) = 1 + 2R_o \cdot \mathcal{L}^{-1}\left[\frac{1}{p\left(\sqrt{1+p}-1\right)}\right] \quad (25)$$

Partial but Meaningful Analytic Solution

For the unsteady interfacial concentration in the constant flux mode

$$\phi_m(\tau) = 1 + 2R_o \left[\tau + \left(\tau + \tfrac{1}{2}\right)\operatorname{erf}\left(\sqrt{\tau}\right) + \sqrt{\frac{\tau}{\pi}}e^{-\tau}\right] \quad (26)$$

Special cases include

- A zero rejection $R_o \to 0$

$$\lim_{R_o \to 0} \phi_m(\tau) = 1 \quad (27)$$

- The perfect rejection $R_o \to 1$

$$\lim_{R_o \to 1} \phi_m(\tau) = (1 + 2\tau)\left[1 + \operatorname{erf}\left(\sqrt{\tau}\right)\right] + 2\sqrt{\frac{\tau}{\pi}}e^{-\tau} \quad (28)$$

- For a large $\tau \gg 1$, the **asymptotic** form of ϕ_m appears

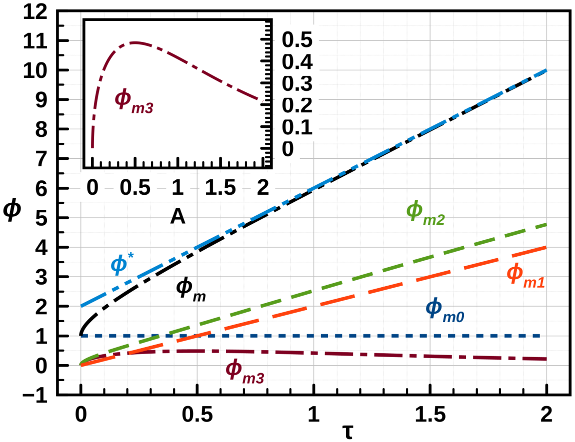
$$\phi^* = \lim_{\tau \gg 1} \phi_m \to 1 + R_o(1 + 4\tau) \quad (29)$$

Solution Components a Perfect Observed Rejection $R_o = 1$

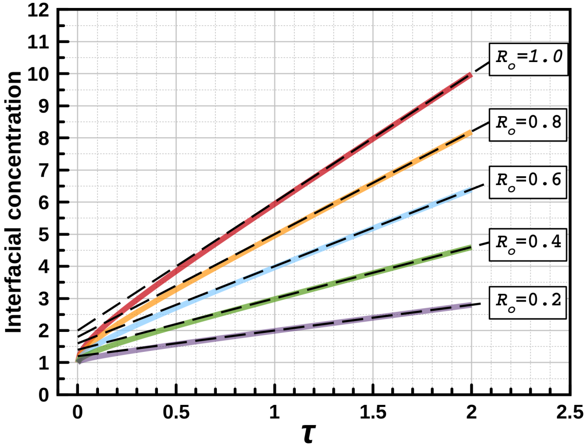
$$
\begin{aligned}
\phi_m(\tau) &= \phi_{m0} + \phi_{m1} + \phi_{m2} + \phi_{m3} \\
\phi_{m0} &= 1 \\
\phi_{m1} &= 2R_o \cdot \tau \\
\phi_{m2} &= 2R_o \cdot \left(\tau + \tfrac{1}{2}\right)\operatorname{erf}\left(\sqrt{\tau}\right) \\
\phi_{m3} &= 2R_o \cdot \sqrt{\frac{\tau}{\pi}} e^{-\tau}
\end{aligned}
$$

which are compared with

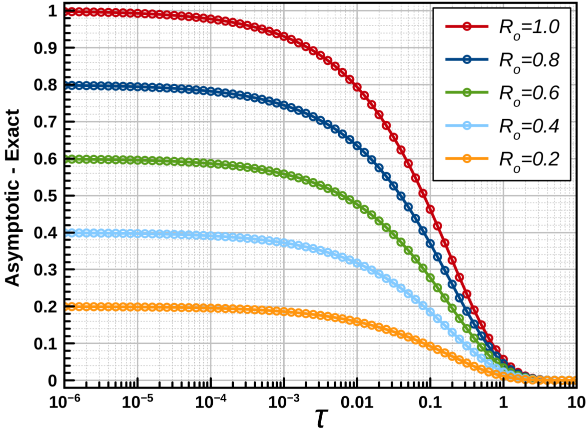
$$\phi^* = \lim_{\tau \gg 1} \phi_m \to 1 + R_o(1 + 4\tau)$$



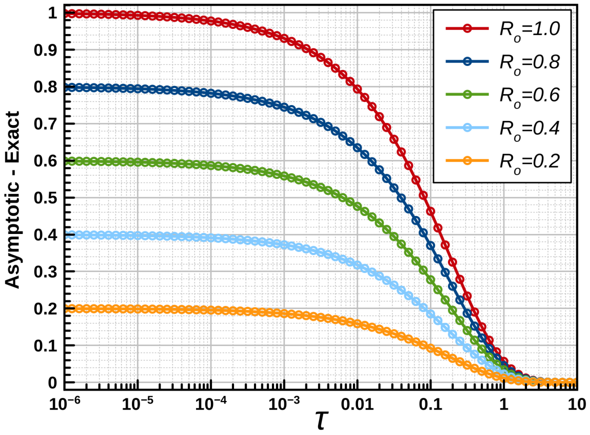
Effects of the Observed Rejection Ratio



Differences between the asymptotic and exact solutions



Differences between the asymptotic and exact solutions



With R_o from 0.2 to 1.0.

Errors between ϕ^* and ϕ_m

- 15.3% at $\tau = 0.5$
- 5.68% at $\tau = 1.$

where $\tau = 1$ means the elapsed filtration time equal to the reference time, i.e., $t = T \simeq 20\,\mathrm{min}.$

Decline of Interfacial Concentration after Pressure Release, $\tau > \tau_s$

The governing equation for $\lambda = 0$ is

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2} \quad \text{for} \quad \tau \geq \tau_s \qquad (30)$$

The far–field BC is kept valid

$$\phi\left(\eta \to \infty, \tau \geq \tau_s\right) = 1 \qquad (31)$$

but the interfacial condition is changed from Robin to the Neumann BC, such as

$$\left[\frac{\partial \phi_m}{\partial \eta}\right]_{\eta=0} = 0 \qquad (32)$$

Therefore,

The analytic solution for $\tau > \tau_s$ is obtained with $\beta = (\phi_{\max} - 1)/R\tau$, such as

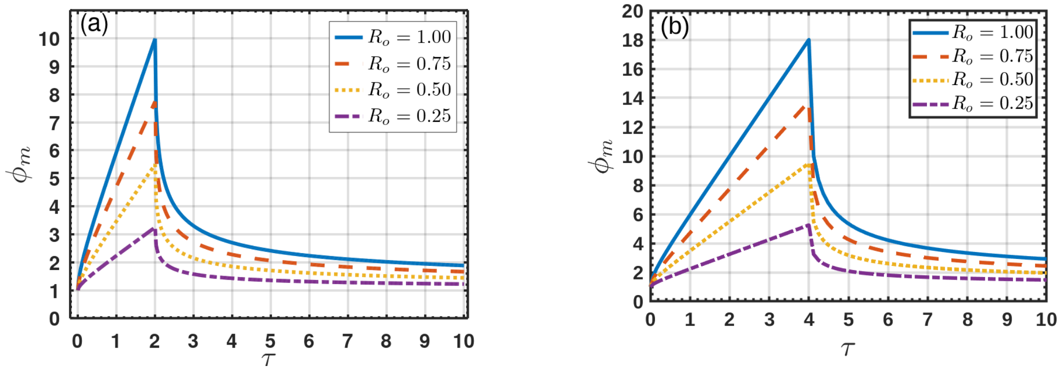
$$\varphi\left(\tau\right) = 1 + \left(\phi_{\max} - 1\right) e^{\beta^2(\tau - \tau_s)} \operatorname{erfc}\left(\beta\sqrt{\tau - \tau_s}\right) \qquad (33)$$

compared with one in the pressing phase

$$\phi_m\left(\tau\right) = 1 + 2R_o\left[\tau + \left(\tau + \tfrac{1}{2}\right)\operatorname{erf}\left(\sqrt{\tau}\right) + \sqrt{\frac{\tau}{\pi}} e^{-\tau}\right]$$

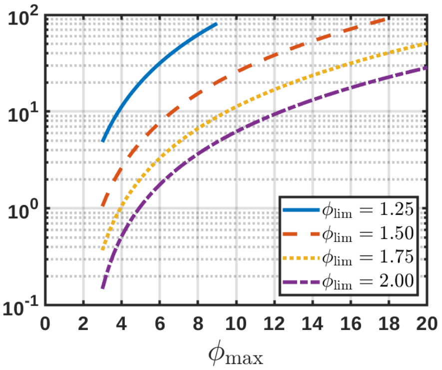
Interfacial concentration growth and decline with τ_s

(a) for stopping time $\tau_s = 2$ (b) for stopping time $\tau_s = 4$.

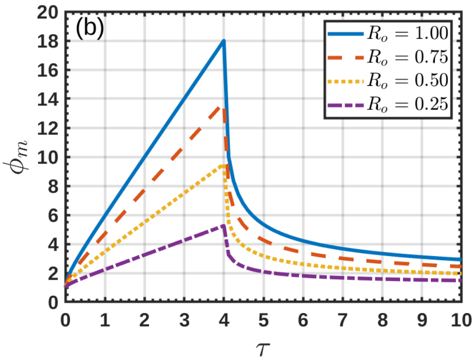


Duration $\Delta\tau$ to reach the limiting concentration $\phi_{\lim}$

Elapsed time regardless of R_o



(b) for stopping time $\tau_s = 4$.



$R_0 = 0.5$, $\phi_{\max} \simeq 9$, $\phi_{\lim} \simeq 2$: $\Delta\tau = 6$

Observed vs. Intrinsic Rejection Ratios

Previous theoretical work

- Dresner (1964) for the constant permeate flux with $R_o = R_i = 1.0$.
- R.J. Raridon, et al.(1966) for the constant permeate flux with finite R_i, reformulated in this work

$$\psi_m \left(\tau, R_i \to 1^-\right) = \quad \left(1 + 2R_i^2 \tau\right) \quad \left[1 + \mathrm{erf}\left(\sqrt{\tau}\right)\right] + 2\sqrt{\frac{\tau}{\pi}} e^{-\tau} \tag{34}$$

- As compared to our work with finite R_o,

$$\phi_m\left(\tau\right) = 1 + 2R_o\left[\tau + \left(\tau + \tfrac{1}{2}\right)\mathrm{erf}\left(\sqrt{\tau}\right) + \sqrt{\frac{\tau}{\pi}} e^{-\tau}\right] \tag{35}$$

Observed vs. Intrinsic Rejection Ratios

The difference between ϕ_m and ψ_m for high rejection and large $\tau\,(>1)$:

$$\phi_m - \psi_m \simeq 4\left(R_o - R_i^2\right)\tau \to 0 \qquad (36)$$

This logically gives us a novel relationship derived purely analytically without experiments and numerical analysis.

$$R_i \simeq \sqrt{R_o} \qquad (37)$$

- Any experimental verifications?

Observed vs. Intrinsic Rejection Ratios

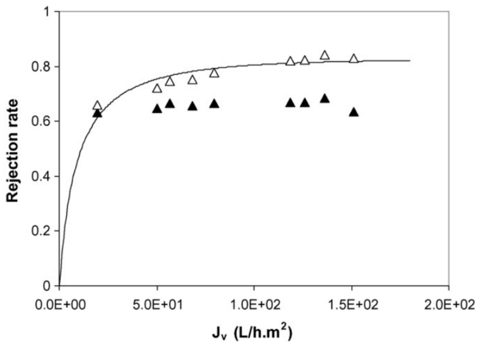


Fig. 3. Variation of the observed (▲) and intrinsic (△) rejection rates with permeate flux (J_V); glucose solution at 2 g L^{-1}; calculations of R_{int} have been carried out by considering the fluid density equals 1, $\eta = 0.89 \times 10^{-3}$ Pa s and $D_{glucose,\infty} = 6.9 \times 10^{-10}$ m^2/s [34].

- Bouranene et al. (2008)'s experimental observations include $R_o \simeq 0.64$ and $R_i \simeq 0.82$ for two cases of removing glucose solution of 2.9g/L and cobalt solution of 0.10g/L using polyamide NF membranes.
- Their rejection values support the theoretical approximation, such as

$$\frac{R_i}{\sqrt{R_o}} = \frac{0.82}{\sqrt{0.64}} \simeq 1.025 \qquad (38)$$

Concluding Remarks

1. This work developed a complete analytic solution for the interfacial concentration as a function of filtration time τ and observed rejection ratio R_o.

$$\phi(\tau,\eta=0)=\phi_m(\tau)=1+2R_o\left[\tau+\left(\tau+\tfrac{1}{2}\right)\mathrm{erf}\left(\sqrt{\tau}\right)+\sqrt{\frac{\tau}{\pi}}e^{-\tau}\right]$$

2. A specific relationship between the observed rejection and intrinsic rejection is mathematically derived and experimentally (indirectly) verified.

$$R_i \simeq \sqrt{R_o}$$

3. A full solution of $\phi(\tau,\eta)$ is coming for the constant-flux dead-end filtration. For the constant-pressure operation, only series analytic solutions can be available.

Other Analytical Work

Eur. Phys. J. E **24**, 331–341 (2007)
DOI 10.1140/epje/i2007-10244-x

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Permeate flux inflection due to concentration polarization in crossflow membrane filtration: A novel analytic approach

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Other Analytical Work

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